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ON THE CONTROL OF SYSTEMS WITH ELASTIC ELEMENTS *

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Certain problems are examined of the control of the plane rotation of a rigid body (a flywheel) by means of an elastic rod (a shaft) at whose end a bounded moment of forces is applied. The main control requirement is the lessening or extinguishing of the flywheel's elastic vibrations at the end of the rotation process. The case of large torsional rigidity of the shaft, being of practical importance, is investigated and estimates of the residual vibrations are presented. The principal purpose of the investigation is to find certain simple practical solutions of control problems for mechanical systems containing elastic components, using forces concentrated at an endpoint. Problems of optimization and control of vibratory processes in mechanical systems by using lumped and distributed forces, leading to the consideration of hyperbolic equations, have been investigated in /1— 6/ and elsewhere.

1. Statement of the problem.

We examine a homogeneous elastic rod (shaft) of constant cross-section, which can be turned around an axis OX, being the axis of symmetry. A controlling force moment is applied at the left end of the shaft ($x = 0$) and an absolutely rigid body, a flywheel, is attached at the right end ($x = l$) (see Fig.1). Notation: $\varphi(t, x)$ is the angle of rotation of the shaft's cross-section at a distance x from point O at instant t; I is the linear density of the shaft's moment of inertia relative to the axis of symmetry; J is the moment of inertia of the flywheel rigidly attached to the shaft; l is the shaft's length; $c = \text{const}$ is the torsional rigidity /7,8/; $M(t)$ is the controlling moment. It is assumed that at the initial instant $t = 0$ the system is at an equilibrium position and at rest. Its motion is described by the differential equation with initial and boundary conditions /1,8/

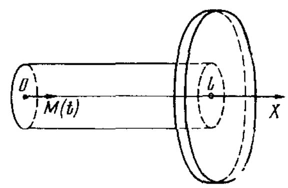

Fig.1

$$I\frac{\partial^2\varphi}{\partial t^2} = c\frac{\partial^2\varphi}{\partial x^2}, \quad \varphi(0, x) = \frac{\partial\varphi(0, x)}{\partial t} = 0 \tag{1.1}$$

$$c\frac{\partial\varphi(t, l)}{\partial x} = -J\frac{\partial^2\varphi(t, l)}{\partial t^2}, \quad c\frac{\partial\varphi(t, 0)}{\partial x} = -M(t), \quad |M| \leqslant M_0$$

The problem is posed of choosing a controlling moment $M(t)$, bounded as in (1.1), which permits the system to be turned through a specified angle φ_* in a finite time T, all vibrations having been annulled, i.e., of bringing the system (1.1) to the state

$$\varphi(T, x) = \varphi_*, \quad \partial\varphi(T, x) / \partial t = 0, \quad x \in [0, l] \tag{1.2}$$

The instant T in (1.2) is not fixed in advance but is found during the solving of the problem. It can be determined by prescribing additional requirements, for example, optimality in accord with some control performance criterion /1—4/, or by other means /6/.

To investigate the control problem (1.1), (1.2) posed it is convenient to pass to dimensionless variables and parameters by using the relations

$$t' = \left(\frac{c}{J_0 l}\right)^{1/2} t, \quad x' = \frac{x}{l}, \quad \varphi' = \frac{c}{M_0 l}\varphi, \quad \varphi_*' = \frac{c}{M_0 l}\varphi_* \tag{1.3}$$

$$M'(t) = \frac{M(t)}{M_0}$$

Here J_0 is some parameter characterizing the system's moment of inertia and equal, for instance, to J or to Il or to their sum. We note that (1.1) contains two typical quantities having the dimensionality of frequency. The quantity $(c / (J_0 l))^{1/2}$ characterizes the frequency of the system's natural oscillations, while $(M_0 / J_0)^{1/2}$ characterizes the angular rotation velocity. It is further assumed that their ratio is a quantity of the order of unity. Then from (1.3) it follows that the coefficient of φ, equal to $c / M_0 l$, also is of the order of unity. As a result of changes (1.3) the control problem (1.1), (1.2) is reduced to (henceforth the primes are omitted)

$$\varepsilon\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial x^2}, \quad \varphi(0, x) = \frac{\partial\varphi(0, x)}{\partial t} = 0 \quad \frac{\partial\varphi(t, 1)}{\partial x} = -\mu\frac{\partial^2\varphi(t, 1)}{\partial t^2}, \quad \frac{\partial\varphi(t, 0)}{\partial x} = -M(t), \quad |M(t)| \leqslant 1 \tag{1.4}$$

$$\varphi(T, x) = \varphi_*, \quad \partial\varphi(T, x) / \partial t = 0, \quad x \in [0, 1]$$

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Here $\varepsilon = IlJ_0^{-1}$ and $\mu = JJ_0^{-1}$ are numerical parameters characterizing the relative mangitudes of the moments of inertia of the shaft and of the flywheel, respectively. A control problem of form (1.4) also describes the translational movement of an elastic rod (a distributed spring) in the direction of its longitudinal axis. It is assumed that a lumped force is applied to one of the rod's ends and that an absolutely rigid body is inflexibly attached to the other. In this case $\varphi(t, x)$ is the absolute displacement of the rod's cross-section, x is the cross-section's relative coordinate, t is time, $M(t)$ is the magnitude-bounded force along the axis OX, I is the rod's line density, J is the rigid body's mass, c is Young's modulus /7,8/, φ_* is the prescribed distance by which the system as a whole must be moved with vibrations extinguished. The parameters J_0, ε, μ have similar meaning.

2. Turning of an elastic shaft.

We consider first the case when the shaft does not load the flywheel, i.e., $J = 0$. This corresponds to the zero value of parameter μ. Setting $J_0 = Il$ ($\varepsilon = 1$), we construct a solution φ° of boundary-value problem (1.4) for $\mu = 0$. This problem contains an inhomogeneous boundary condition when $x = 0$. To construct a solution of the inhomogeneous boundary-value problem we use the approach suggested in /9/. It is well known that the eigenfunctions $\varphi_n(x)$ corresponding to a homogeneous boundary-value problem have the form $\varphi_n(x) = \cos \pi n x$ ($n = 0, 1, 2, \ldots$) /8/. We seek the solution of problem (1.4) as the series

$$\varphi^\circ(t, x) = \sum_{n=0}^{\infty} C_n(t) \cos \pi n x \tag{2.1}$$

Because the system of functions $\cos \pi n x$, $x \in [0, 1]$ is orthogonal, the coefficients of series (2.1) are

$$C_0(t) = \int_0^1 \varphi^\circ(t, x)\, dx, \quad C_n(t) = 2\int_0^1 \varphi^\circ(t, x) \cos \pi n x\, dx, \quad n \geqslant 1 \tag{2.2}$$

Having multiplied both sides of differential equation (1.4) by $\cos \pi n x$ and integrated with respect to x, $x \in [0, 1]$, after simple manipulations we obtain the relations

$$\frac{\partial^2}{\partial t^2} \int_0^1 \varphi^\circ(t, x) \cos \pi n x\, dx = \frac{\partial \varphi^\circ(t, 1)}{\partial x} \cos \pi n - \frac{\partial \varphi^\circ(t, 0)}{\partial x} - \pi^2 n^2 \int_0^1 \varphi^\circ(t, x) \cos \pi n x\, dx$$

Allowing for the boundary conditions and for expressions (2.2), from the relations derived we find the equations for the coefficients of series (2.1) and the corresponding initial conditions

$$C_0^{\cdot\cdot} = M(t), \quad C_n^{\cdot\cdot} + \pi^2 n^2 C_n = 2M(t), \quad n \geqslant 1$$
$$C_n(0) = C_n^{\cdot}(0) = 0, \quad n = 0, 1, \ldots \tag{2.3}$$

Using (2.1) and (2.3), for the variable φ we obtain the desired solution of boundary-value problem (1.4) with $\mu = 0$

$$\varphi^\circ(t, x) = \int_0^t (t - \tau) M(\tau)\, d\tau + \sum_{n=1}^{\infty} \left[\frac{2}{\pi n} \int_0^t \sin \pi n (t - \tau) M(\tau)\, d\tau\right] \cos \pi n x \tag{2.4}$$

If function $M(t)$ is piecewise continuous and smooth on the intervals of continuity, series (2.4) converges absolutely and uniformly. The first derivatives of function φ° are mean-square convergent /8/. We note that the first summand in (2.4) is a solution of the ordinary differential equation $\varphi^{\cdot\cdot} = M(t)$ with conditions $\varphi(0) = \varphi^{\cdot}(0) = 0$. It describes the motion of an absolutely rigid body with moment of inertia $Il = 1$, rotating around the axis OX under the force moment $M(t)$. Thus, the first summand corresponds to the turning of the shaft as a whole. The second summand (the series) describes the elastic torsional vibrations.

Let us consider a control $M(t)$ of the form

$$M(t) = \alpha \operatorname{sign}(T/2 - t), \quad t \in [0, T]$$
$$M(t) \equiv 0, \quad t \overline{\in} [0, T], \quad 0 < \alpha \leqslant 1 \tag{2.5}$$

Here α and T are constants; their values are determined below. We substitute $M(t)$ from (2.5) into (2.4) and integrate. For φ° and $\partial\varphi^\circ / \partial t$ at instant $t = T$ we obtain the expressions

$$\varphi^\circ(T, x) = \frac{\alpha T^2}{4} + \frac{2\alpha}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(2 \cos \frac{\pi}{2} nT - 1 - \cos \pi n T\right) \cos \pi n x \tag{2.6}$$
$$\frac{\partial \varphi^\circ(T, x)}{\partial t} = \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left(\sin \pi n T - 2 \sin \frac{\pi}{2} nT\right) \cos \pi n x$$

From (2.6) it follows that when $T = 4m$, where m is an integer, the series equal zero and the identities

$$\varphi(T, x) = \alpha T^2 / 4, \quad \partial\varphi(T, x) / \partial t = 0, \quad x \in [0, 1] \tag{2.7}$$

hold. In order to satisfy the final conditions from (1.4) the parameter α and the integer m in (2.7) must be chosen so as to fulfil the relations

$$4\alpha m^2 = \varphi_*, \quad \alpha \leqslant 1 \tag{2.8}$$

From (2.8) it follows that $m \geqslant {}^1/_2\sqrt{\varphi_*}$, while the set of integers m is determined by

$$m = m_i = {}^1/_2\sqrt{\varphi_*} + i - 1, \; i = 1, 2, \ldots, \quad \text{if } \{{}^1/_2\sqrt{\varphi_*}\} = 0$$
$$m = m_i = [{}^1/_2\sqrt{\varphi_*}] + i, \; i = 1, 2, \ldots, \quad \text{if } \{{}^1/_2\sqrt{\varphi_*}\} \neq 0 \tag{2.9}$$

Here $\{k\}$ and $[k]$ denote the fractional and the integer parts of number k, respectively. The values of control parameter α are determined from (2.8) and (2.9)

$$\alpha = {}^1/_4\varphi_* m_i^{-2}, \quad i = 1, 2, \ldots \tag{2.10}$$

Thus, from (2.7)-(2.10) it follows that by using control (2.5) we can in finite time $T = 4m_i$ turn a homogeneous shaft through a prescribed angle φ_*, having stopped all its vibrations. We remark that an analogous means of constructing admissible control (2.5) with one switching point was used in /10/ in the study of a pendulum displacement problem.

From (2.9) and (2.10) it follows that the value $m = m_*$, corresponding to the least possible shaft turning time with annihilation of vibrations when using controls of form (2.5), equals $m^* = {}^1/_2\sqrt{\varphi_*}$ if $\{{}^1/_2\sqrt{\varphi_*}\} = 0$ and $m_* = [{}^1/_2\sqrt{\varphi_*}] + 1$ if $\{{}^1/_2\sqrt{\varphi_*}\} \neq 0$. The corresponding value of turning time T_* is determined by the relations

$$T_* = 4m_* = \begin{cases} 2\sqrt{\varphi_*}, & \{{}^1/_2\sqrt{\varphi_*}\} = 0 \\ 4[{}^1/_2\sqrt{\varphi_*}] + 4, & \{{}^1/_2\sqrt{\varphi_*}\} \neq 0 \end{cases} \tag{2.11}$$

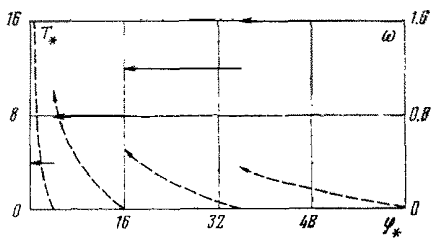

Fig.2

In Fig.2 the solid lines show the graph of T_* as a function of shaft turning angle φ_*. We observe that according to (2.11) the minimum dimensionless time T_* is not less than four for an arbitrarily small $\varphi_* > 0$. An analogous property was noted in /11/ in the investigation of the finite control problem. The dashed lines show the graph of the function $\omega(\varphi_*) = [T_*(\varphi_*) - 2\sqrt{\varphi_*}] / (2\sqrt{\varphi_*})$, exhibiting the relative difference between the turning time T_* for the elastic shaft and the least possible time $2\sqrt{\varphi_*}$ viz., the time-optimal fast for the turning of a rigid body with a moment of inertia equal to Il.

If using (1.3) we go back to the dimensional variables, expression (2.11) for the least turning time T_* under control (2.5) becomes

$$T_* = \begin{cases} 2(Il\varphi_* / M_0)^{1/2}, & \{{}^1/_2(c\varphi_* / (M_0 l))^{1/2}\} = 0 \\ 4\sqrt{Il^2/c}\,([{}^1/_2(c\varphi_* / (M_0 l))^{1/2}] + 1), & \{{}^1/_2(c\varphi_* / (M_0 l))^{1/2}\} \neq 0 \end{cases} \tag{2.12}$$

For the exact solution constructed above for the control problem it proves possible to study T_* from (2.12) for arbitrary values of the parameters. In particular, $T_* \to 2(Il\varphi_* / M_0)^{1/2}$ as $c \to \infty$. This signifies that the minimum time T_* for turning an elastic shaft through any specified angle φ_*, with vibrations damped out, tends to a minimum as the rigidity c grows unboundedly, i.e., to the turning time of an absolutely rigid body.

3. Investigation of the control problem in the case of regular perturbations. Let us study the control problem (1.1), (1.2) under the assumption that the flywheel's moment of inertia is small: $J \ll Il$ and that its influence can be treated as a perturbation. Then, setting $J_0 = Il$, we arrive at a control problem (1.4) in which $\varepsilon = 1$ and $\mu \ll 1$. In this case, under a control (2.5) for which the parameter α is defined by (2.10), small residual vibrations $\Delta\varphi = \varphi - \varphi^\circ$ appear when $t > T_*$. To estimate $\Delta\varphi$ we construct the solution of problem (1.4) in the form

$$\varphi = \varphi^\circ(t, x) + \mu\varphi^1, \quad \Delta\varphi = \mu\varphi^1 \tag{3.1}$$

Here $\varphi^\circ(t, x)$ is solution (2.4) of problem (1.4) with $\mu = 0$ under a control $M(t)$ of form (2.5). Substituting (3.1) into (1.4) and retaining terms of the first order of smallness in μ, we obtain a boundary-value problem for function φ^1

$$\frac{\partial^2\varphi^1}{\partial t^2} = \frac{\partial^2\varphi^1}{\partial x^2}, \quad \varphi^1(0, x) = \frac{\partial\varphi^1(0, x)}{\partial t} = 0 \tag{3.2}$$

$$\frac{\partial\varphi^1(t, 0)}{\partial x} = 0, \quad \frac{\partial\varphi^1(t, 1)}{\partial x} = -F(t), \quad F(t) = \frac{\partial^2\varphi^\circ(t, 1)}{\partial t^2} \tag{3.3}$$

No final conditions are imposed on φ^1 since we are studying the problem of estimating φ^1 for $t > T$. We observe that boundary-value problem (3.2), (3.3) is analogous to problem (1.4). Using the approach in /9/, applied in Sect.2, we obtain the required expression of form (2.4) for the variable $\varphi^1(t, x)$

$$\varphi^1(t, x) = -\int_0^t (t-\tau) F(\tau)\, d\tau - 2\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{\pi n}\int_0^t \sin\pi n(t-\tau) F(\tau)\, d\tau\right]\cos\pi n x \tag{3.4}$$

Differentiating (3.4) with respect to t, we obtain

$$\frac{\partial\varphi^1(t, x)}{\partial t} = -\int_0^t F(\tau)\, d\tau - 2\sum_{n=1}^{\infty}(-1)^n\left[\int_0^t \cos\pi n(t-\tau) F(\tau)\, d\tau\right]\cos\pi n x \tag{3.5}$$

Let us estimate expressions (3.4) and (3.5) for $t > T$, where $T = 4m_i$. Since $F(t) \equiv 0$ for $t > T$, the integration with respect to τ in (3.4) and (3.5) is from 0 to T. After the substitution of $F(t)$ from (3.3) into (3.4) and integration, the first summand becomes (for $t > T$)

$$-\int_0^t (t-\tau) F(\tau)\, d\tau = -\int_0^T (t-\tau) F(\tau)\, d\tau = -\varphi^\circ|_0^T = -\frac{\alpha T^2}{4} \tag{3.6}$$

By direct integration we find that

$$\int_0^T \sin\pi n(t-\tau) F(\tau)\, d\tau = 0, \quad t \geqslant T \tag{3.7}$$

Thus, the second summand (the series) in (3.4) equals zero. As a result it follows from (3.4)-(3.7) that

$$\varphi^1(t, x) = -\alpha T^2/4 = \text{const}, \quad \partial\varphi^1(t, x)/\partial t = 0, \quad t \geqslant T$$

Consequently, when $t \geqslant T$

$$\varphi(t, x) = {}^1/_4\alpha T^2(1-\mu) + O(\mu^2), \quad \varphi^{\cdot}(t, x) = O(\mu^2)$$

4. Investigation of a singularly perturbed control problem. Let us consider the limiting case of practical importance when the flywheel's moment of inertia J is much greater than that of the elastic shaft when the latter's torsional rigidity is sufficiently large, i.e., $J \gg Il$ and $c \sim M_0 l$. Such a situation obtains in many engineering systems. Setting $J_0 = J$ ($\mu = 1$, $\varepsilon \ll 1$), we reduce boundary-value problem (1.4) to the form

$$\varepsilon\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial x^2}, \quad \varepsilon = \frac{Il}{J}, \quad \varphi(0, x) = \frac{\partial\varphi(0, x)}{\partial t} = 0 \tag{4.1}$$

$$\frac{\partial\varphi(t, 1)}{\partial x} = -\frac{\partial^2\varphi(t, 1)}{\partial t^2}, \quad \frac{\partial\varphi(t, 0)}{\partial x} = -M(t)$$

Direct verification convinces us that the function

$$\varphi = \varphi_0(t, x) = M(t)(1-x) + \int_0^t (t-\tau) M(\tau)\, d\tau \tag{4.2}$$

is a solution of (4.1) when $\varepsilon = 0$ and satisfies the boundary conditions at $x = 0$ and $x = 1$. We note that at $x = 1$ the function φ_0 from (4.2) satisfies the initial conditions at $t = 0$ as well. Consequently, the variation of the rigid body's angular coordinate is determined by expression (4.2) with $x = 1$

$$\varphi_0(t, 1) = \int_0^t (t-\tau) M(\tau)\, d\tau \tag{4.3}$$

Function $\varphi_0(t, 1)$ coincides with the solution of equation $\varphi^{\cdot\cdot}(t, 1) = M(t)$ under zero initial conditions, describing the rigid body's rotation around a fixed axis OX under the action of the force moment M. Thus, when $\varepsilon = 0$ the motion of the flywheel located at the end of an elastic rod coincides with rotation in the case when the force moment is applied directly to the body.

This fact has a simple physical interpretation. An infinitely large velocity of propagation of the elastic forces along the shaft corresponds to the parameter value $\varepsilon = 0$. From (4.3) it follows that to solve the problem of turning the flywheel through a prescribed angle φ_* when $\varepsilon = 0$ it is sufficient to select a control $M_*(t)$ not exceeding unity in absolute value, such that the equalities

$$\varphi_0(T, 1) = \int_0^T (T-\tau) M(\tau)\, d\tau = \varphi_*, \quad \frac{\partial \varphi_0(T, 1)}{\partial t} = \int_0^T M(\tau)\, d\tau = 0 \tag{4.4}$$

are fulfilled at some instant T.

Let $M_*(t)$ be a function satisfying (4.4) and identically equalling zero outside the interval $[0, T]$. Let us investigate the motion of system (4.1) with $\varepsilon \neq 0$ and control $M = M_*(t)$. We seek the solution of boundary-value problem (4.1) in the form

$$\varphi = \varphi_0(t, x) + \Phi \tag{4.5}$$

where $\varphi_0(t, x)$ is determined by (4.2). Substituting (4.5) into (4.1), we obtain a boundary-value problem for Φ

$$\varepsilon \frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^2} - \varepsilon M_*^{\cdot\cdot}(t)(1-x) - \varepsilon M_*(t) \tag{4.6}$$

$$\Phi(0, x) = -M_*(0)(1-x), \quad \partial\Phi(0, x)/\partial t = -M_*^{\cdot}(0)(1-x)$$

$$\partial\Phi(t, 1)/\partial x = -\partial^2\Phi(t, 1)/\partial t^2, \quad \partial\Phi(t, 0)/\partial x = 0$$

Passing in (4.6) to a new variable ψ

$$\psi = \Phi - {}^1\!/_2 \varepsilon (1+\varepsilon)^{-1} x^2 M_*(t) \tag{4.7}$$

we obtain a boundary-value problem with homogeneous boundary conditions

$$\frac{\partial^2\psi}{\partial t^2} = \frac{1}{\varepsilon}\frac{\partial^2\psi}{\partial x^2} - M_*^{\cdot\cdot}\frac{\varepsilon x^2}{2(1+\varepsilon)} - M_*(1-x) - M_*\frac{\varepsilon}{1+\varepsilon} \tag{4.8}$$

$$\psi(0, x) = -M_*(0)\left[(1-x) + \frac{\varepsilon x^2}{2(1+\varepsilon)}\right],$$

$$\frac{\partial\psi(0, x)}{\partial t} = -M_*^{\cdot}(0)\left[(1-x) + \frac{\varepsilon x^2}{2(1+\varepsilon)}\right]$$

$$\partial\psi(t, 1)/\partial x = -\varepsilon^{-1}\partial^2\psi(t, 1)/\partial x^2, \quad \partial\psi(t, 0)/\partial x = 0$$

Using the Fourier method /8/, for $\psi(t, x)$ we obtain the expression

$$\psi(t, x) = -M_*(t)\left[(1-x) + \frac{1}{2}\varepsilon(1+\varepsilon)^{-1}x^2\right] - \frac{\varepsilon}{1+\varepsilon}\int_0^t M_*(\tau)(t-\tau)\, d\tau + \sum_{n=1} \frac{2\sqrt{\varepsilon}\cos\lambda_n x}{\lambda_n(\varepsilon+\cos^2\lambda_n)}\int_0^t \sin\frac{\lambda_n}{\sqrt{\varepsilon}}(t-\tau) M_*(\tau)\, d\tau \tag{4.9}$$

Here λ_n are the eigenvalues of the corresponding boundary-value problem (4.8), i.e., are the roots of the equation

$$\varepsilon \operatorname{tg}\lambda = -\lambda, \quad \lambda = \lambda_n,\ n = 1, 2, \ldots \quad (\lambda_n^{-1} = O(n^{-1}),\ n \to \infty) \tag{4.10}$$

On the basis of (4.7) we obtain a solution of boundary-value problem (4.6)

$$\Phi(t, x) = M_*(t)(x-1) - \frac{\varepsilon}{1+\varepsilon}\int_0^t (t-\tau) M_*(\tau)\, d\tau + \sum_{n=1}^{\infty} \frac{2\sqrt{\varepsilon} I_n^{\varepsilon}(t)\cos\lambda_n x}{\lambda_n(\varepsilon+\cos^2\lambda_n)}, \quad I_n^{\varepsilon}(t) = \int_0^t \sin\frac{\lambda_n}{\sqrt{\varepsilon}}(t-\tau) M_*(\tau)\, d\tau \tag{4.11}$$

According to (4.11) with respect to time t derivative of Φ equals

$$\frac{\partial\Phi(t, x)}{\partial t} = M_*^{\cdot}(t)(x-1) - \frac{\varepsilon}{1+\varepsilon}\int_0^t M_*(\tau)\, d\tau + \sum_{=1}^{\infty} \frac{2J_n^{\varepsilon}(t)\cos\lambda_n x}{\varepsilon+\cos^2\lambda_n}, \quad J_n^{\varepsilon}(t) = \int_0^t \cos\frac{\lambda_n}{\sqrt{\varepsilon}}(t-\tau) M_*(\tau)\, d\tau \tag{4.12}$$

Let us estimate the integrals $I_n^{\varepsilon}(t)$ and $J_n^{\varepsilon}(t)$ in (4.11) and (4.12). We assume that $M_*(t)$ is piecewise-continuous and piecewise continuously differentiable on the set of continuity. Let t_i $(i = 1, 2, \ldots, N-1)$ be the points of discontinuity of $M_*(t)$ or of its derivative on the interval $(0, T)$, $t_0 = 0$, $t_N = T$. Then the coefficients $I_n^{\varepsilon}(t)$ can be given as

$$I_n^{\varepsilon}(t) = \sum_{i=0}^{r(t)-1}\int_{t_i}^{t_{i+1}} \sin\frac{\lambda_n}{\sqrt{\varepsilon}}(t-\tau) M_*(\tau)\, d\tau + \tag{4.13}$$

$$\int_{t_r}^{t} \sin \frac{\lambda_n}{\sqrt{\varepsilon}}(t-\tau) M_*(\tau)\, d\tau$$

Here $r(t)$ denotes the maximum index on the points t_i of discontinuity of $M_*(t)$ and $M_*^{\cdot}(t)$, where $t_i \leqslant t$. Integrating each summand in (4.13) by parts, for $t > T$ we obtain the estimate

$$|I_n^{\varepsilon}(t)| \leqslant \frac{\sqrt{\varepsilon}}{\lambda_n} A, \quad A = \sum_{i=0}^{N-1} \{|M_*(t_{i+1}-0)| + |M_*(t_i+0)|\} + \sup_{t\in[0,T]\setminus\Omega} |M_*^{\cdot}(t)|\, T \tag{4.14}$$

Here we have reckoned that $M_*(t) \equiv 0$ when $t > T$; by Ω we have denoted the set of points of discontinuity of $M_*(t)$ and $M_*^{\cdot}(t)$. The estimate for coefficients $J_n^{\varepsilon}(t)$

$$|J_n^{\varepsilon}(t)| \leqslant \sqrt{\varepsilon} A \lambda_n^{-1} \tag{4.15}$$

is proved similarly to (4.14).

Proceeding from relations (4.11)— (4.15) let us estimate the values of the function $\Phi(t, 1)$ and of the derivative $\partial\Phi(t, 1) / \partial t$ when $t \geqslant 0$. From (4.11) and (4.14) we have an estimate for the coefficients in the series in (4.11) when $x = 1$

$$\left| \frac{2\sqrt{\varepsilon} I_n^{\varepsilon}(t) \cos \lambda_n}{\lambda_n(\varepsilon + \cos^2 \lambda_n)} \right| = \left| \frac{2\varepsilon \sqrt{\varepsilon} I_n^{\varepsilon}(t) \sin \lambda_n}{\lambda_n^2(\varepsilon + \cos^2 \lambda_n)} \right| \leqslant \frac{2\varepsilon A}{\lambda_n^3} \tag{4.16}$$

From the property of convergence of series with terms $\lambda_n^{-3} = O(n^{-3})$ (see /12/) follows

$$\Phi(t, 1) = O(\varepsilon), \quad t \in [0, \infty) \tag{4.17}$$

By an estimate analogous to (4.16) it can be proved that

$$\partial\Phi(t, 1) / \partial t = O(\sqrt{\varepsilon}), \quad t \in [0, \infty) \tag{4.18}$$

According to (4.5) the function $\Phi(t, x)$ describes the difference of the motion of a system controlled by moment $M_*(t)$ for an arbitrary value of parameter ε $(0 < \varepsilon \ll 1)$ from the motion when $\varepsilon = 0$. The function $\Phi(t, 1)$ describes the difference of the motion of the rigid body (the flywheel), allowing for a small inertia of the rod (the shaft), from the motion with an inertialess shaft. Estimates (4.17)and (4.18) show that the difference is a quantity of the order of ε in the turning angle and a quantity of the order of $\sqrt{\varepsilon}$ in the turning velocity on the whole time interval $t \in [0, \infty)$. This enables us to conclude that if the elastic rod's moment of inertia is small in comparison with that of the rigid body, the control law can be calculated by setting $\varepsilon = 0$. The error thus obtained is small (in the sense of (4.17) and (4.18)).

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